THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH4240 Stochastic Processes, 2023-24 Term 2

Test 2

Answer all FOUR questions (Total points: 100 points). Give adequate explanation and justification for all your computations and observations, and write your proofs and reasonings in a clear, rigorous and complete way as much as you can.

1. (35 points) Let $\{X_n\}_{n\geq 0}$ be a Markov chain on $\{0,1,2,3,4\}$ with transition matrix

$$P = \begin{array}{c} \\ \\ \end{array}\begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & \frac{1}{3} & \frac{2}{3} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{4} & \frac{3}{4} \\ 0 & 0 & 0 & \frac{1}{4} & \frac{3}{4} \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}.$$

   (a) Explain why the stationary distribution π uniquely exists without solving the equation $\pi P = \pi$.
   (b) Suppose you know $\pi = \left(\frac{1}{3}, \frac{1}{9}, \frac{2}{9}, \frac{1}{12}, \frac{1}{4}\right)$ is the unique distribution of the chain. For each state x, find $m_x := E_x(T_x)$, denoting the mean returning time to x for the chain starting from x.
   (c) Does the limit $\lim\limits_{n\to\infty} P^n$ exist? Justify your answer with a reason. Hint: Compute P^2, P^3 and P^4.

Solution:

(a) Notice that $\{0\} \to \{1,2\} \to \{3,4\} \to \{0\}$ in the transition matrix and hence the chain is irreducible. Since the state space is finite, the chain is positive recurrent and consequently there exists a unique stationary distribution π.

(b) Recall that the unique stationary distribution π of an irreducible positive recurrent Markov chain is given by

$$\pi(x) = \frac{1}{m_x}, \quad x \in S.$$

Hence we have

$$m_0 = 3, \quad m_1 = 9, \quad m_2 = \frac{9}{2}, \quad m_3 = 12, \quad m_4 = 4.$$

(c) We have

$$P^2 = \begin{pmatrix} 0 & 0 & 0 & \frac{1}{4} & \frac{3}{4} \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & \frac{2}{3} & 0 & 0 \\ 0 & \frac{1}{3} & \frac{2}{3} & 0 & 0 \end{pmatrix}, \quad P^3 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & \frac{2}{3} & 0 & 0 \\ 0 & \frac{1}{3} & \frac{2}{3} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{4} & \frac{3}{4} \\ 0 & 0 & 0 & \frac{1}{4} & \frac{3}{4} \end{pmatrix}, \quad P^4 = \begin{pmatrix} 0 & \frac{1}{3} & \frac{2}{3} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{4} & \frac{3}{4} \\ 0 & 0 & 0 & \frac{1}{4} & \frac{3}{4} \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} = P.$$

We can see this Markov chain has period 3 and $\lim\limits_{n\to\infty} P^n$ does not exist.

2. (35 points) Let $\{X_n\}_{n\geq 0}$ be a Markov chain on state space $\{1,2,3,4,5\}$ with transition matrix

$$P = \begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 & 5 \end{matrix} \\ \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{5} & \frac{4}{5} \\ 0 & 0 & 0 & \frac{2}{5} & \frac{3}{5} \\ 1 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \end{bmatrix} \end{array}.$$

(a) Determine if this chain is irreducible and if it is aperiodic. Give detailed reasonings to your answer.

(b) Solve $\pi P = \pi$ to find any possible stationary distribution.

(c) Let $X_0 = 1$, compute the expected number of steps until it is in state 1 again.

Solution:

(a) Denote $i \to j$ if $P(i,j) > 0$, where P is the transition probability. Note that in this matrix

$$1 \to 2 \to 4 \to 1 \to 3 \to 5 \to 1,$$

the chain is irreducible.

As $P(5,5) = \frac{1}{2} > 0$, the irreducible chain is aperiodic.

(b) It suffices to compute some distribution π such that $\pi P = \pi$, $\sum_i \pi(i) = 1$ and $\pi(i) \geq 0$ for all $i \in S$. To have $\pi P = \pi$, we need to solve $(P^T - I)\pi^T = 0$, where I is the identity matrix. Solve it, the solution set is

$$\left\{ \left(\frac{5t}{7}, \frac{5t}{14}, \frac{5t}{14}, \frac{3t}{14}, t \right) : t \in \mathbb{R} \right\}.$$

$\sum_i \pi(i) = 1$ gives $t = \frac{14}{37}$. Set $\pi = (10/37, 5/37, 5/37, 3/37, 14/37)$.

By direct computation, one verifies that $\sum_i \pi(i) = 1$ and $\pi(i) \geq 0$ for all $i \in S$. The Markov chain is irreducible and finite, so this π is the unique stationary distribution.

(c) The Markov chain is irreducible with finite state space, then

$$E_1(T_1) = m_1 = \frac{1}{\pi(1)} = 3.7.$$

3. (20 points) Consider the Markov chain with state space $S = \{0, 1, 2, \ldots\}$ and transition probabilities:

$$P(n, n+1) = p, \quad P(n, 0) = 1 - p, \quad n \in S,$$

with $0 < p < 1$.

(a) Solve $\pi P = \pi$ to find all possible stationary distributions. Give detailed computations.

(b) Determine if the chain is positive recurrent. Give full reasonings to your answer.

Solution:

(a) For all $x, y \in S$,

$$\rho_{xy} > P(x,0)P^y(0,y) = P(x,0)P(0,1)^y = (1-p)p^y > 0.$$

Then the Markov chain is irreducible. From $\pi P = \pi$ and $\sum_{x=0}^{\infty} \pi(x) = 1$, we have that

$$\begin{aligned}
\pi(0) &= \sum_{x=0}^{\infty} \pi(x)P(x,0) = (1-p)\sum_{x=0}^{\infty} \pi(x) = 1-p \\
\pi(1) &= \pi(0)P(0,1) = (1-p)p \\
\pi(2) &= \pi(1)P(1,2) = (1-p)p^2 \\
&\vdots
\end{aligned}$$

By induction, $\pi(x) = (1-p)p^x, x \geq 0$. Then by geometric series, we can check that π satisfies both $\sum_{x=0}^{\infty} \pi(x) = 1$ and $\pi(x) = \sum_{y=0}^{\infty} \pi(y)P(y,x), x \geq 0$. Hence the stationary distribution is $\pi = (1-p) \cdot (1, p, p^2, \ldots)$.

(b) As the irreducible Markov chain has a stationary distribution, the chain is positive recurrent.

4. (10 points) Let $\{X_n\}_{n\geq 0}$ be a Markov chain with state space S and transition matrix $P = [P(x,y)]_{x,y\in S}$, where S can be finite or countably infinite.

(a) Let $y \in S$ be either a null recurrent state or a transient state in S. Explain that

$$\lim_{n\to\infty} E_x\left(\frac{N_n(y)}{n}\right) = 0, \quad \forall\, x \in S,$$

where $N_n(y)$ denotes the number of visits to y in n steps and $E_x(\cdot)$ denotes the expectation for the chain initially starting from x. Give reasons.

(b) Use (a) to prove that if there is no positive recurrent state then there is no stationary distribution for the chain.

Solution:

(a) If y is transient, since

$$\lim_{n\to\infty} E_x(N_n(y)) = E_x(N(y)) = \frac{\rho_{xy}}{1-\rho_{yy}} < \infty,$$

we have that

$$\lim_{n\to\infty} E_x\left(\frac{N_n(y)}{n}\right) = \lim_{n\to\infty} \frac{E_x(N_n(y))}{n} = 0.$$

If y is null-recurrent i.e. $m_y = E_y(T_y) = \infty$, then we have

$$\lim_{n\to\infty} E_x\left(\frac{N_n(y)}{n}\right) = \frac{\rho_{xy}}{m_y} = 0.$$

(b) Suppose there is a stationary distribution π. Note that

$$\begin{aligned}
\pi(y) &= \sum_{x \in S} \pi(x) P^m(x, y) \\
&= \sum_{x \in S} \pi(x) \frac{\sum_{m=1}^{n} P^m(x, y)}{n} \\
&= \sum_{x \in S} \pi(x) E_x \left(\frac{N_n(y)}{n} \right)
\end{aligned}$$

Since there is no positive recurrent state, each state is either transient or null-recurrent. However, according to the bounded convergence theorem, if $n \to \infty$, then we would have

$$\begin{aligned}
\pi(y) &= \lim_{n \to \infty} \sum_{x \in S} \pi(x) E_x \left(\frac{N_n(y)}{n} \right) \\
&= \sum_{x \in S} \pi(x) \lim_{n \to \infty} E_x \left(\frac{N_n(y)}{n} \right) \\
&= \sum_{x \in S} \pi(x) \cdot 0 \quad \text{by part (a)} \\
&= 0 \quad \forall\, y \in S.
\end{aligned}$$

This is a contradiction to the fact that π is a stationary distribution.

—THE END—